SOME RATIONAL SUBVARIETIES OF MODULI SPACES OF STABLE VECTOR BUNDLES

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Abstract. Let X be a smooth complex irreducible projective variety of dimension $n \geq 2$ and H be an ample line bundle on X. In this paper, we construct families of μ_H-stable vector bundles on X having fixed determinant and rank r, which are generated by $r+1$ global sections, parametrized by Grassmanian varieties. This gives into the corresponding moduli spaces special subvarieties birational to Grassmannian.

Introduction

The notion of μ-stability for vector bundles on curves was introduced by Mumford, and subsequently extended to higher-dimensional varieties by the foundational works of Takemoto, Gieseker and Maruyama. In particular, Maruyama proved the existence of coarse moduli spaces parametrising isomorphism classes of μ_H-stable vector bundles with respect to an ample polarisation H, on a smooth projective variety (see [Mar77]).

While the case of curves is nowadays well understood, the situation in higher dimension remains considerably less developed. In particular, there are no general results ensuring the non-emptyness of these moduli spaces. For this reason, explicit constructions of families of μ-stable vector bundles dominating particular subvarieties of these moduli spaces seem to be of significant interest.

Let X be a smooth complex irreducible projective variety of dimension $n \geq 2$ and let L be a non-trivial globally generated line bundle on X. In this paper, our aim is to produce families of vector bundles on X with rank $r \geq 2$ and determinant L, which are generated by $r+1$ global sections and are μ_H-stable with respect to an ample line bundle H on X. Moreover, these families give rise to subvarieties in the corresponding moduli spaces which are birational to a Grassmannian variety.

Our construction starts as follows. Let $W \subset H^0(L)$ be a $(r+1)$-dimensional subspace such that the evaluation map of global sections $W \otimes \mathcal{O}_X \to L$ is a surjective map of vector bundles on X. Denote by $M_{W,L}$ its kernel; it is then a vector bundle on X of rank r and determinant L^{-1}. Its dual is a vector bundle E_W too, with rank r, determinant L, and Chern classes $\underline{c} = (c_1(L), \ldots, c_1(L)^n)$ (see Lemma 2.6), which fit into the following exact sequence:

$$0 \to L^{-1} \to W \otimes \mathcal{O}_X \to E_W \to 0.$$

If $M_{W,L}$ is μ_H semistable for an ample line bundle H on X, then so is E_W and it is generated by $r+1$ global sections.

Vector bundles of the form $M_{W,F}$ (denoted as M_F in the complete case $W = H^0(F)$), arising as kernels of evaluation map of globally generated vector bundles F, on a smooth variety, are

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known in literature as *kernel bundles, dual span bundles and sygyzy bundles*. Their stability has been extensively studied. For a smooth curve of genus $g \geq 2$, the theory is well developed at least for the complete case (see, for example, the results in [But94], [Mis08], [EL92], [CH25], [BBPN08]); there are also some results in the case of singular curves (see for example [BF20]). In higher dimension, only partial results are available, mainly in the complete case and for line bundles (see [Fle84] and [EL13] and [Cam12]).
Our strategy for proving the stability of $M_{W,L}$ consists in reducing the problem to the stability of kernel bundles on smooth curves. More precisely, let H be an ample line bundle on X and assume that there exists a smooth curve $C \subset X$ of genus $g \geq 2$, given as a complete intersection of divisors of $|H|$, such that the restriction map of global section $H^0(X, L) \to H^0(C, L_{|C})$ is surjective. We can prove that the restriction of $M_{W,L}$ to C is a kernel bundle on C and its stability implies μ_H-stability of $M_{W,L}$. Stability on the curve C is ensured by requiring suitable numerical assumptions on the degree of $L_{|C}$. Specifically, our result holds whenever either our conditions or those established in [Mis08] are satisfied.
We will say that the data (X, L, H, r) is *admissible* if the above mentioned assumptions are satisfied (c.f. Definition 2.1). We denote by $\mathcal{M}^s_H(r, L, \underline{c})$ the moduli space parametrizing μ_H-stable vector bundles with rank r, determinant L, and Chern classes $\underline{c}$ depending on L (c.f. Definition 2.12). Our main result is the following (see Theorem 2.14):

Theorem. *Let (X, L, H, r) an admissible collection, then the moduli space $\mathcal{M}^s_H(r, L, \underline{c})$ is non-empty and it contains a subvariety birational to the Grassmannian variety* $\mathrm{Gr}(r+1, H^0(L))$.

This provides, in arbitrary dimension, a systematic method to construct globally generated μ_H-stable vector bundles with prescribed determinant and Chern classes.

In the second part of the paper, we specialise to algebraic surfaces, and we investigate the scope of our construction through a series of examples. We exhibit admissible collections with surfaces for each Kodaira dimensions $\kappa(S) \in \{-\infty, 0, 1, 2\}$. Of particular interest is the case of K3 surfaces. Indeed, when S is a K3 surface and H is an ample primitive line bundle on S, the subvariety arising from our construction turns out to be a Lagrangian subvariety of the moduli space, provided the latter is a smooth irreducible symplectic variety (see Theorem 3.11 and Remark 3.12).

1. Notations and preliminary results

1.1. **Moduli spaces of stable sheaves.** Let X be a smooth irreducible projective complex variety of dimension $n \geq 2$ and H an ample line bundle on X. We will need to deal with moduli spaces parametrising (H-stable) vector bundles on X. In this section, we recall some well-known results on this topic. Our main reference is [HL10]. To begin with, we recall that - unlike in the case of curves - obtaining a projective moduli space requires us to include torsion-free sheaves on X.

Let E be a non-trivial torsion-free sheaf on X. There exist a non empty open subset $U \subseteq X$ such that $E|_U$ is a vector bundle. Then $\mathrm{rk}(E)$ is defined as the rank of $E|_U$. When the pair (X, H) is fixed, one can define the μ_H-semistability and H-semistability through the H-slope μ_H of E and its reduced Hilbert polynomial, respectively. We recall that the μ_H-slope is

$$\mu_H(E) = \frac{c_1(E) \cdot H^{n-1}}{\mathrm{rk}(E)}$$

whereas the reduced Hilbert polynomial is, up to a positive constant which depends only on the pair (X, H),

$$p_H(E,k) = \frac{\chi(E \otimes H^{\otimes k})}{\mathrm{rk}(E)}.$$

A torsion-free sheaf E is called *μ_H-semistable* if for any non zero subsheaf $F \subset E$ with $\mathrm{rk}(F) < \mathrm{rk}(E)$ we have $\mu_H(F) \leq \mu_H(E)$, it is said μ_H-stable it the strict inequality holds.
The sheaf E is said H-semistable if for any non-zero subsheaf $F \subset E$ we have $p_H(F,k) \leq p_H(E,k)$ for $k >> 0$ and it is said H-stable if the strict inequality holds for any proper subsheaf F. One has the following chain of implications:

$$E \text{ is } \mu_H\text{-stable} \implies E \text{ is } H\text{-stable} \implies E \text{ is } H\text{-semistable} \implies E \text{ is } \mu_H\text{-semistable}.$$

Any line bundle on X is μ_H-stable. Taking duals and tensoring by line bundles preserve both H-semistability and H-stability. Moreover, the sum of two μ_H-semistable vector bundles is μ_H-semistable if and only if they have the same H-slope.

Any H-semistable torsion free sheaf E admits a Jordan-Holder fibration

$$(JH) \quad 0 = E_0 \subset E_1 \subset \cdots \subset E_l = E$$

with $\mathrm{gr}(E_i) = \frac{E_i}{E_{i-1}}$ which is H-stable with reduced Hilbert polynomial $p_H(E,k)$. So one can define the graded object $\mathrm{gr}(E) = \bigoplus \mathrm{gr}(E_i)$. Two H-semistable torsion-free sheaves are said S-equivalent if they have isomorphic graded objects.

Let $P(k) \in \mathbb{Q}[k]$ be a polynomial of degree n, and denote by $\mathcal{M}_H(P)$ the moduli space parametrizing S-equivalence classes of H-semistable torsion free sheaves E on X with Hilbert polinomial (with respect to the polarization H) given by $P_H(E) = P$. The existence of this moduli space is guaranteed, for example, by [HL10, Theorem 3.4.4]. $\mathcal{M}_H(P)$ is a projective scheme, containing as an open subscheme the moduli space $\mathcal{M}^s_H(P)$ parametrizing isomorphism classes of μ_H-stable vector bundles. Finally, if $\underline{c} = (c_1, c_2, \cdots, c_n)$ with $c_i \in H^{2i}(X,\mathbb{Z})$, $\mathcal{M}^s_H(P)$ is a disjoint union of schemes $\mathcal{M}^s_H(r,\underline{c})$, where $\mathcal{M}^s_H(r,\underline{c})$ is the moduli space of μ_H-stable vector bundles on X of rank r with Chern classes $(c_1, c_2, \cdots, c_n)$ up to numerical equivalence (see [Mar77]). We recall that by Bogomolov's inequality if E is a torsion free μ_H-semistable sheaf of rank r on X we have

$$\Delta_H(E) = (2rc_2(E) - (r-1)c_1(E)^2) \cdot H^{n-2} \geq 0; \tag{1}$$

this was proved by Bogomolov [Bog78] for surfaces and generalized to higher dimensional smooth projective varieties using Mumford-Mehta-Ramanathan restriction theorem [MR84].

Let $L \in \mathrm{Pic}(X)$ be a line bundle. We denote by $\mathcal{M}^s_H(r,L,\underline{c})$ the moduli space of μ_H-stable vector bundles E with $\det E = L$ and Chern classes $c_i(E) = c_i$, $i = 2, \cdots n$. This is simply the fiber at L of the morphism $\det\colon \mathcal{M}^s_H(r,\underline{c}) \to \mathrm{Pic}(X)$ which sends $[E]$ to its determinant $\det(E)$. Finally, we recall the following properties concerning the infinitesimal structure of these moduli spaces. Assume that there exists $[E] \in \mathcal{M}^s_H(r,L,\underline{c})$, which is the isomorphism class of a μ_H-stable vector bundle. Then

$$T_{[E]}(\mathcal{M}^s_H(r,L,\underline{c})) \simeq \mathrm{Ext}^1(E,E)_0,$$

$$\dim \mathrm{Ext}^1(E,E)_0 - \dim \mathrm{Ext}^2(E,E)_0 \leq \dim_{[E]} \mathcal{M}^s_H(r,L,\underline{c}) \leq \dim \mathrm{Ext}^1(E,E)_0,$$

where $\mathrm{Ext}^i(E,E)_0$ is the kernel of the map $h^i(tr)\colon \mathrm{Ext}^i(E,E) \to H^i(\mathcal{O}_X)$ induced by the trace map $tr\colon \mathrm{End}(E) \to \mathcal{O}_X$, see [HL10]. If $\mathrm{Ext}^2(E,E)_0 = 0$, then the moduli space is smooth at the point $[E]$.

In particular, if S is a smooth surface and L is a line bundle on S, then $\underline{c}$ is identified by the choice of c_2 so we can write $\mathcal{M}^s_H(r, L, \underline{c}) = \mathcal{M}^s_H(r, L, c_2)$, for brevity. If $[E]$ is the isomorphism class of a μ_H-stable vector bundle in $\mathcal{M}^s_H(r, L, c_2)$, then

$$\operatorname{edim}(\mathcal{M}^s_H(r, L, c_2)) := \dim \operatorname{Ext}^1(E, E)_0 - \dim \operatorname{Ext}^2(E, E)_0 = \\ = 2rc_2 - (r-1)L^2 - (r^2-1)\chi(\mathcal{O}_S), \tag{2}$$

and it is the *expected dimension* [HL10, Def. 4.5.6] of the moduli space $\mathcal{M}^s_H(r, L, c_2)$ at $[E]$.

Finally, we define the discriminant

$$\Delta(r, L, c_2) := 2rc_2 - (r-1)L^2. \tag{3}$$

By Bogomolov's inequality the moduli space $\mathcal{M}^s_H(r, L, c_2)$ is empty if $\Delta(r, L, c_2)$ is negative. If $\Delta(r, L, c_2) >> 0$, the moduli space $\mathcal{M}^s_H(r, L, c_2)$ is a normal, generically smooth, irreducible quasi-projective variety of the expected dimension; this result is due to many authors, see [MRO09] for a survey. Moreover, when S is a K3 surface, by the seminal works of Mukai (see [Muk84] [Muk87]), then $\mathcal{M}^s_H(r, L, c_2)$, if nonempty, is a smooth quasi-projective variety of the expected dimension which has a simplectic structure.

1.2. **Globally generated vector bundles of rank r with $r+1$ global sections.** Let (X, H) a pair as above. Let E be a vector bundle on X with rank $r \geq 2$. The *evaluation map* of global sections of E associated to E is

$$ev_E \colon H^0(E) \otimes \mathcal{O}_X \to E, \quad s \mapsto s(x). \tag{4}$$

We can construct the maps

$$\wedge^r(ev_E) \colon (\wedge^r H^0(E)) \otimes \mathcal{O}_X \to \wedge^r E, \quad s_1 \wedge s_2 \wedge \cdots \wedge s_r \to s_1(x) \wedge s_2(x) \wedge \cdots \wedge s_r(x); \tag{5}$$

and the *determinant map* of E, namely

$$d_E = H^0(\wedge^r ev_E) : \wedge^r H^0(E) \to H^0(\det(E)), \tag{6}$$

i.e. the map induced by $\wedge^r(ev_E)$ on global sections.

We recall that E is said *globally generated* if the evaluation map ev_E is surjective. In this case, as the trivial vector bundle $H^0(E) \otimes \mathcal{O}_X$ is μ_H-semistable, for any ample line bundle H on X, we obtain that $\mu_H(E) \geq 0$.

Now we assume that E is globally generated and $h^0(E) = r+1$. We set $L = \det(E)$ for brevity, and we consider the exact sequence

$$0 \to L^* \to H^0(E) \otimes \mathcal{O}_X \xrightarrow{ev_E} E \to 0 \tag{7}$$

and its dual

$$0 \to E^* \xrightarrow{ev_E^*} H^0(E)^* \otimes \mathcal{O}_X \xrightarrow{\gamma} L \to 0 \tag{8}$$

where γ is the dual of the inclusion $L^* \hookrightarrow H^0(E) \otimes \mathcal{O}_X$ composed via the canonical isomorphism $L \simeq (L^*)^*$. The following is a technical result we will use in the sequel.

Proposition 1.1. *Let E be a globally generated vector bundle of rank $r \geq 2$ with $h^0(E) = r+1$. If E is μ_H-stable, for a ample line bundle H on X, then*

(a) d_E is injective;
(b) $\operatorname{Im}(d_E)$ is equal to $\operatorname{Im}(H^0(\gamma))$.

Proof. (a) As the sequence (7) is an exact sequence of vector bundles, we have an induced sequence

(9) $$0 \to \ker(\wedge^r ev_E) \to \bigwedge^r H^0(E) \otimes \mathcal{O}_X \xrightarrow{\wedge^r ev_E} L \to 0.$$

and a canonical isomorphism

$$\ker(\wedge^r ev_E) \simeq L^{-1} \otimes \bigwedge^{r-1} E \simeq E^*$$

which follows from the isomorphism $\bigwedge^r E = \det(E) = L$ (see [Har77, Chapter II.5]). Since E is μ_H−stable and $\mu_H(E) \geq 0$, one can prove that $\mathrm{Hom}(E, \mathcal{O}_X) \simeq H^0(E^*) = 0$. So we can conclude that the map induced in cohomology

$$d_E = H^0(\wedge^r ev_E) : \bigwedge^r H^0(E) \to H^0(L)$$

is injective.

(b) Being $h^0(E) = r + 1$, we have the canonical isomorphism

(10) $$\eta : \wedge^r H^0(E) \to H^0(E)^* \qquad \omega \mapsto \{s \mapsto \omega \wedge s\}$$

and then an isomorphism $\eta' = \eta \otimes \mathrm{id}_{\mathcal{O}_X} : \bigwedge^r H^0(E) \otimes \mathcal{O}_X \to H^0(E)^* \otimes \mathcal{O}_X$. Consider the following exact sequences:

(11) $$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{Ker}(\wedge^r ev_E) & \longrightarrow & \bigwedge^r H^0(E) \otimes \mathcal{O}_X & \xrightarrow{\wedge^r ev_E} & L & \longrightarrow & 0 \\ & & & & \downarrow{\scriptstyle \eta'} & & & & \\ 0 & \longrightarrow & \mathrm{Ker}(\gamma) & \longrightarrow & H^0(E)^* \otimes \mathcal{O}_X & \xrightarrow{\gamma} & L & \longrightarrow & 0 \end{array}$$

We claim that $\eta'(\mathrm{Ker}(\wedge^r ev_E)) = \mathrm{Ker}(\gamma)$. Recall that, for all $x \in X$, one has

$$L_x^* \simeq \mathrm{Ker}(ev_E)_x = H^0(E \otimes \mathcal{I}_x) \otimes \mathcal{O}_x = \langle \tau_x \rangle \otimes \mathcal{O}_x,$$

by the short exact sequence (7). Under our assumptions we have

$$\mathrm{Ker}(\wedge^r ev_E)_x = \{\omega \in \wedge^r H^0(E) \,|\, \omega \wedge \tau_x = 0\} \otimes \mathcal{O}_x.$$

On the other hand, one has

$$\gamma_x : H^0(E)^* \otimes \mathcal{O}_x \to L_x$$

is the map induced by the restriction of forms on $H^0(E)$ to $H^0(E \otimes \mathcal{I}_x)$. Then

$$\mathrm{Ker}(\gamma)_x = \{\varphi \in H^0(E)^* \,|\, H^0(E \otimes \mathcal{I}_x) \subseteq \mathrm{Ker}(\varphi)\} \otimes \mathcal{O}_x.$$

Notice that if $\omega \in \wedge^r H^0(E)$, then $\tau_x \in \mathrm{Ker}(\eta(\omega)) \iff \omega \wedge \tau_x = 0$ so

$$\eta'(\mathrm{Ker}(\wedge^r ev_E)) = \mathrm{Ker}(\gamma)$$

as claimed. Then there exists $\alpha : L \to L$ which makes commutative the diagram on the right in (11). Actually, being η' an isomorphism, by Snake Lemma, α is an isomorphism too. Since L is a line bundle, this is an homothety.

Finally, we have a commutative diagram

(12) $$\begin{array}{ccc} \bigwedge^r H^0(E) & \xrightarrow{d_E} & H^0(L) \\ {\scriptstyle \eta}\downarrow & & \downarrow{\scriptstyle H^0(\alpha)} \\ H^0(E)^* & \xrightarrow[H^0(\gamma)]{} & H^0(L) \end{array}$$

As $H^0(\alpha) = \lambda \cdot \mathrm{id}_{H^0(L)}$, this concludes the proof.

□

2. Main construction

In this section, we consider a smooth complex projective variety X of dimension $n \geq 2$. We recall that if L is a line bundle on X and W is a (non-trivial) subspace of $H^0(L)$ one denotes by

$$\varphi_{|W|}\colon X \dashrightarrow \mathbb{P}(W)^* \qquad p \mapsto \{s \in W \mid s(p) = 0\}$$

the usual map induced by global sections of W. We will simply write φ_L instead of $\varphi_{H^0(L)}$, for brevity.

Definition 2.1. *Consider the collection (X, L, H, r) where X is a smooth complex projective variety of dimension $n \geq 2$, L and H are line bundles on X and r is an integer with $r \geq 2$, satisfying the following conditions:*

- A_1**:** *H is ample and there exists a smooth irreducible curve C of genus $g \geq 2$ which is complete intersection of divisors in $|H|$;*
- A_2**:** *L is big, nef, globally generated, $r \geq \dim(\varphi_L(X))$ and the restriction map of global sections $\rho\colon H^0(L) \to H^0(L|_C)$ is surjective;*
- A_3**:** *If we set $d = \deg(L|_C)$, then either*
  - $A_3(1)$**:** *$d = rg + 1$ or;*
  - $A_3(2)$**:** *$r + g + 1 \leq d \leq \min(2r, r + 2g)$ and if $d = 2r$, C is not hyperelliptic.*

We will say that (X, L, H, r) is admissible if assumptions A_1, A_2 and A_3 hold.

Remark 2.2. *We stress that, as we are assuming $r \geq 2$ and $g \geq 2$, it does not exists d that satisfies the two numerical conditions in $A_3(1)$ and $A_3(2)$ simultaneously.*

Remark 2.3. *As will be clear in the sequel, the curve C will only be auxiliary to the construction and the results will not depend on the specific choice of C. For this reason, C is not part of the building data (X, L, H, r).*

For any $k \geq 1$, $\mathrm{Gr}(k, H^0(L))$ will denote the Grassmannian variety parametrizing k-dimensional linear susbspaces of $H^0(L)$.

Lemma 2.4. *Let (X, L, H, r) an admissible collection and let C and ρ be as in the Definition 2.1. Then ρ induces a rational surjective map*

$$R_C\colon \ \mathrm{Gr}(r + 1, H^0(L)) \dashrightarrow \mathrm{Gr}(r + 1, H^0(L|_C)), \quad W \mapsto \rho(W).$$

Moreover, for W general in $\mathrm{Gr}(r+1, H^0(L))$, $|W|$ and $|\rho(W)|$ are base points free linear systems.

Proof. By A_3 it follows that $\deg(L|_C) = d \geq 2g + 1$, so we have $h^1(L|_C) = 0$ and $h^0(L|_C) = d + 1 - g$. By A_2, the restriction map $\rho : H^0(L) \to H^0(L|_C)$ is surjective, so

$$h^0(L) \geq h^0(L|_C) = d + 1 - g > r + 1 \tag{13}$$

by assumptions A_1 and A_3.

In particular, $\mathrm{Gr}(r + 1, H^0(L))$ and $\mathrm{Gr}(r + 1, H^0(L|_C))$ are both not empty and

$$\mathrm{codim}_{H^0(L)}(\mathrm{Ker}(\rho)) = h^0(L|_C) > r + 1.$$

Hence, for $W \in \mathrm{Gr}(r + 1, H^0(L))$ general, we have that $\mathrm{Ker}(\rho) \cap W = \{0\}$ so $\rho|_W\colon W \to \rho(W)$ is an isomorphism.

This defines the rational map R_C which is also surjective since ρ is surjective and by A_1.

We claim now that there exists a non-empty open subset of $\mathrm{Gr}(r+1, H^0(L))$ which parametrises base point free linear systems. Recall that there exists a canonical isomorphism

$$\alpha : \mathrm{Gr}(r+1, H^0(L)) \to \mathrm{Gr}(h^0(L) - (r+1), H^0(L)^*)$$

which associates to W the kernel Λ of the dual of the inclusion $W \hookrightarrow H^0(L)$. Moreover, if $\Lambda = \alpha(W)$, the projection $\pi : \mathbb{P}(H^0(L))^* \dashrightarrow \mathbb{P}W^* \simeq \mathbb{P}^r$ from $\mathbb{P}(\Lambda)$ fits into the diagram

$$\begin{array}{ccc} X & \xrightarrow{\varphi_L} & \mathbb{P}H^0(L)^* \\ & \searrow^{\varphi_{|W|}} & \downarrow \pi \\ & & \mathbb{P}W^* \end{array} \tag{14}$$

As L is globally generated, one has that $\varphi_{|W|}$ is a morphism if and only if $\varphi_L(X) \cap \mathbb{P}(\Lambda)$ is empty. Actually, this occurs for general $\Lambda \in \mathrm{Gr}(h^0(L) - r - 1, H^0(L)^*)$ since $\mathrm{codim}_{\mathbb{P}(H^0(L))^*}(\mathbb{P}(\Lambda)) = r + 1 \geq \dim(\varphi_L(X)) + 1$ by A_2. Hence, for general W, one has that $|W|$ is base point free. The elements of the linear system $|\rho(W)|$ are the intersection of the divisors in $|W|$ with C, so $|\rho(W)|$ is base point free too. □

Remark 2.5. *By Lemma 2.4, since as observed in the above proof, one has $h^0(L|_C) = d+1-g$, it follows that:*

$$\dim \mathrm{Gr}(r+1, H^0(L)) \geq \dim \mathrm{Gr}(r+1, H^0(L|_C)) = (r+1)(d-g-r).$$

If assumptions $A_3(1)$ holds, one has $\dim \mathrm{Gr}(r+1, H^0(L|_C)) = (r^2-1)(g-1)$.

Let $W \in \mathrm{Gr}(r+1, H^0(L))$ such that $|W|$ is base point free. Hence, the evaluation map ev_W associated to W is surjective and its kernel is a locally free sheaf on X of rank r which fits in the following exact sequence

$$0 \to \ker(ev_W) \to W \otimes \mathcal{O}_X \xrightarrow{ev_W} L \to 0, \tag{15}$$

whose dual is

$$0 \to L^* \to W^* \otimes \mathcal{O}_X \to \ker(ev_W)^* \to 0. \tag{16}$$

We define

$$E_W : = \ker(ev_W)^*. \tag{17}$$

Lemma 2.6. *Let $W \in \mathrm{Gr}(r+1, H^0(L))$ such that $|W|$ is base point free. Then E_W is a vector bundle on X with the following properties:*
(a) $\mathrm{rk}\, E_W = r$, $\det(E_W) = L$ *and* $c_k(E_W) = c_1(L)^k$ *for* $k = 1, \ldots, n$*;*
(b) $H^0(E_W) \simeq W^*$ *and* E_W *is globally generated;*
(c) $W = \mathrm{Im}(H^0(\gamma))$ *where* γ *is defined in exact sequence (8);*

Proof. In order to prove claim (a), recall that $\dim(W) = r+1$ so that $\mathrm{rk}\, E_W = r$ and $\det(E_W) = L$ by the exact sequence (16). From the same sequence, one has

$$1 = c(W^* \otimes \mathcal{O}_X) = c(L^*)c(E_W) = (1 - c_1(L))c(E_W) = 1 + \sum_{k=1}^{n} (c_k(E_W) - c_1(L)c_{k-1}(E_W))$$

by Whitney's sum formula. Then, by induction, one has $c_k(E_W) = c_1(L)^k$.
For claim (b), we get the exact sequence

$$0 \to H^0(L^*) \to W^* \to H^0(E_W) \to H^1(L^*) \to \cdots$$

passing to cohomology from the Exact Sequence (16).

Since L is big and nef by A_2 we have $H^q(L^*) = 0$ for $q < n$ by Kawamata-Viehweg vanishing Theorem, which implies $H^0(E_W) \simeq W^*$. Moreover, the composition of the map $W^* \otimes \mathcal{O}_X \to E_W$ in exact sequence (16) with the isomorphism $W^* \otimes \mathcal{O}_X \simeq H^0(E_W) \otimes \mathcal{O}_X$ is actually the evaluation map $ev_{H^0(E_W)}$. This implies that E_W is globally generated.

In order to prove claim (c), start by dualizing Exact sequence (16) and use what we observed in (b). One gets the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & {E_W}^* & \longrightarrow & W^{**} \otimes \mathcal{O}_X & \xrightarrow{\gamma'} & L & \longrightarrow & 0 \\ & & & \searrow^{(ev_{H^0(E)})^*} & \downarrow \simeq & \nearrow_{\gamma} & & & \\ & & & & H^0(E)^* \otimes \mathcal{O}_X & & & & \end{array}$$

where γ is defined in (8) while γ' is ${ev_W}^{**}$ composed via the canonical isomorphism $L \simeq L^{**}$. In particular, passing to cohomology, we have that the images of $H^0(\gamma)$ and $H^0(\gamma')$ coincide. By construction we have $\mathrm{Im}(H^0(\gamma')) = W$.

$\square$

Remark 2.7. *By Lemma 2.6 it follows that*

$$\Delta_H(E_W) = (r+1)c_1(L)^2 H^{n-2} > 0,$$

so the vector bundle E_W satisfies the generalized Bogomolov's necessary condition (see Equation 1) for μ_H-semistability. Actually, we will prove in Proposition 2.10 that E_W is μ_H-stable.

Let C as in Definition 2.1. If $W \in \mathrm{Gr}(r+1, H^0(L))$ is general, by Lemma 2.4 we have that $\rho(W) \in \mathrm{Gr}(r+1, H^0(L|_C))$ and, moreover, $|\rho(W)|$ is base points free. This means that the evaluation map $ev_{\rho(W)} \colon \rho(W) \otimes \mathcal{O}_C \to L|_C$ is surjective. Its kernel is a locally free sheaf on C which fits in the exact sequence

$$0 \to \ker(ev_{\rho(W)}) \to \rho(W) \otimes \mathcal{O}_C \to L|_C \to 0, \tag{18}$$

whose dual is

$$0 \to L^*|_C \to \rho(W)^* \otimes \mathcal{O}_C \to \ker(ev_{\rho(W)})^* \to 0. \tag{19}$$

Then, by construction,

$$E_{\rho(W)} \colon = \ker(ev_{\rho(W)})^* \tag{20}$$

is a vector bundle of rank r, whose determinant is $\det E_{\rho(W)} = L|_C$.

Remark 2.8. *The same argument used in Lemma 2.6 proves that $E_{\rho(W)}$ is globally generated.*

Lemma 2.9. *Let C be as in Definition 2.1. Then, there exists an open dense subset $U_C \subseteq \mathrm{Gr}(r+1, H^0(L))$ such that for any $W \in U_C$, $|W|$ is base points free, $\rho(W) \simeq W$ and $E_{\rho(W)}$ is a stable vector bundle.*

Proof. Let $W \in \mathrm{Gr}(r+1, H^0(L))$ such that $|W|$ is base points free and $\rho(W) \simeq W$. We distinguish two cases depending on whether $A_3(1)$ or $A_3(2)$ applies.

- Assume that $A_3(1)$ holds. We consider the exact sequence induced by (19), passing to cohomology:

  $$0 \to \rho(W)^* \to H^0(E_{\rho(W)}) \to H^1(L^*|_C) \to \rho(W)^* \otimes H^1(\mathcal{O}_C) \to H^1(E_{\rho(W)}) \to 0.$$

  By $A_3(1)$, one has $\deg(E_{\rho(W)}) = rg + 1$ so that $\chi(E_{\rho(W)}) = r + 1$. This implies that $\rho(W)^* \simeq H^0(E_{\rho(W)})$ if and only if $h^1(E_{\rho(W)}) = 0$. This happens exactly when the map

  $$H^1(L^*|_C) \to \rho(W)^* \otimes H^1(\mathcal{O}_C)$$

is an isomorphism i.e. when the dual map

$$m_{\rho(W)}\colon \rho(W)\otimes H^0(\omega_C)\to H^0(\omega_C\otimes L|_C)$$

is an isomorphism.

We claim that for general W, the multiplication map $m_{\rho(W)}$ is an isomorphism. Since $\deg(L|_C)=rg+1$ we have $\deg(L|_C\otimes\omega_C)=g(r+2)-1\geq 2g-1$ so

$$h^1(L|_C\otimes\omega_C)=0 \qquad \text{and} \qquad h^0(L|_C\otimes\omega_C)=\chi(L|_C\otimes\omega_C)=g(r+1).$$

By [Bri02], one has that $\mu_{W'}$ is surjective for W' general in $\mathrm{Gr}(r+1,H^0(L|_C))$ so

$$V=\{W'\in \mathrm{Gr}(r+1,H^0(L|_C))\,|\,m_{W'}\text{ is an isomorphism}\}$$

is a dense open subset of $\mathrm{Gr}(r+1,H^0(L|_C))$. As R_C is a rastional surjective map, see 2.4, then, $R_C^{-1}(V)$ is a non-empty open subset of $\mathrm{Gr}(r+1,H^0(L))$, and $m_{\rho(W)}$ is an isomorphism for $W\in R_C^{-1}(V)$. Summing up, we concluded that for W general, $H^0(E_{\rho(W)})\simeq\rho(W)^*$.

In order to prove the stability of $E_{\rho(W)}$, we assume that there exists a proper subbundle $G\subset E_{\rho(W)}$ of degree d and rank $s\leq r-1$, such that

$$\mu(G)=\frac{d}{s}\geq\mu(E_{\rho(W)})=\frac{rg+1}{r}.$$

This implies that

$$d\geq sg+1 \qquad \text{and} \qquad \chi(G)\geq sg+1+s(1-g)=s+1.$$

In particular, we have that $h^0(G)\geq s+1$. We claim now that $h^0(G)=s+1$ and that G is globally generated.

Recall that G is a subbundle of $E_{\rho(W)}$, which is globally generated and is such that $h^0(E_{\rho(W)})=r+1$. Assume, by contraddiction, that $h^0(G)>s+1$. Then, the sections of $H^0(G)\subseteq H^0(E_{\rho(W)})$ span a vector bundle G' in $E_{\rho(W)}$ of rank at most s. On the other hand, the remaining sections of $H^0(E_{\rho(W)})$ cannot increase the rank of the spanned vector bundle by more than $h^0(E_{\rho(W)})-h^0(G)<r+1-(s+1)=r-s$. This is impossible as we could have that $E_{\rho(W)}$ is not globally generated: we have that $h^0(G)=s+1$.

In a similar way one proves that G is globally generated (since otherwise we would have points on C where $s+1$ sections would span a vector space of dimension lower than s).

Being $h^0(G)=s+1$, we have

$$s+1-h^1(G)=\chi(G)\geq s+1$$

so that $h^1(G)=0$ and $\deg(G)=sg+1$.

The evaluation maps of G and $E_{\rho(W)}$, fit in the commutative diagram

(21)
$$\begin{array}{ccccccccc}
0 & \longrightarrow & M^* & \longrightarrow & H^0(G)\otimes\mathcal{O}_C & \longrightarrow & G & \longrightarrow & 0\\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & L^* & \longrightarrow & H^0(E_{\rho(W)})\otimes\mathcal{O}_C & \longrightarrow & E_{\rho(W)} & \longrightarrow & 0
\end{array}$$

Since M^* is a subsheaf of L^* we have

$$\deg(M^*)=-\deg(G)=-(sg+1)\leq\deg(L^*)=-(rg+1),$$

and thus $s\geq r$, which is impossible.

- Assume that $A_3(2)$ hold. By the assumptions it follows immediately that $r \geq g+1$ and $d = \deg(L|_C) \geq 2g+2$ so $h^0(L|_C) = d+1-g$. We set
$$c := \operatorname{codim}_{H^0(L|_C)}(\rho(W)) = h^0(L|_C) - (r+1) = d-g-r.$$
By our assumptions on d, it follows that
$$1 \leq c \leq g \qquad \text{and} \qquad d \geq 2g+2c.$$
Then we can apply [Mis08, Theorem 1.3]: $\operatorname{Ker}(ev_V)$ is stable for a general $V \subset H^0(L|_C)$ of codimension c, unless $d = 2g+2c$ and C is hyperelliptic (case which is excluded by $A_3(2)$). Since the rational map R_C is surjective, see lemma 2.4, this gives a non-empty open subset of $\operatorname{Gr}(r+1, H^0(L))$ such that $E_{\rho(W)}$ is stable for any $W \in U_C$.

□

Proposition 2.10. *Let C be as in Definition 2.1 and consider $W \in U_C$. Then*
(a) $E_W|_C \simeq E_{\rho(W)}$;
(b) E_W is μ_H-stable;
(c) the determinant map d_{E_W} is injective and has image W.

Proof. Let $W \in U_C$, E_W defined in Equation (17) and $E_{\rho(W)}$ defined in Equation (20).
In order to prove claim (a), we start by tensoring (16) with $\mathcal{O}_C$ and get the exact sequence

$$0 \to L^*|_C \xrightarrow{(ev_W^*)|_C} W^* \otimes \mathcal{O}_C \to E_W|_C \to 0. \tag{22}$$

By Lemma 2.4 we have that $W \simeq \rho(W)$, hence $W^* \simeq \rho(W)^*$, moreover
$$(ev_W^*)|_C = (ev_W|_C)^*$$
so we get the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & L|_C^* & \xrightarrow{(ev_W|C)^*} & W^* \otimes \mathcal{O}_C & \longrightarrow & E_W|_C & \longrightarrow & 0 \\ & & \downarrow \simeq & & \downarrow \rho^*\otimes \mathrm{id} & & \downarrow \simeq & & \\ 0 & \longrightarrow & (L|_C)^* & \xrightarrow{ev^*_{\rho(W)}} & \rho(W)^* \otimes \mathcal{O}_C & \longrightarrow & E_{\rho(W)} & \longrightarrow & 0. \end{array} \tag{23}$$

To prove claim (b), assume that there exists a subbundle $G \hookrightarrow E_W$ of rank $s \leq r-1$ such that $\mu_H(G) \geq \mu_H(E_W)$. Being C a complete intersection of divisors in $|H|$ one has $\deg(F|_C) = c_1(F) \cdot H^{n-1}$ for any vector bundle F on X. In particular, $G|_C$ and $E_W|_C$ are vector bundles on C which satisfy
$$\mu(G|_C) = \frac{\deg(G|_C)}{s} = \frac{c_1(G) \cdot H^{n-1}}{s} = \mu_H(G)$$
$$\mu(E_W|_C) = \frac{\deg(E_W|_C)}{r} = \frac{c_1(E_W) \cdot H^{n-1}}{r} = \mu_H(E_W)$$
so that $\mu(G|_C) \geq \mu(E_W|_C)$. In particular, $E_W|_C$ is not stable. This is impossible, since $E_W|_C$ is isomorphic to $E_{\rho(W)}$, which is stable by Lemma 2.9.

In order to prove claim (c), since E_W is μ_H-stable, globally generated and has $h^0(E_W) = r+1$ (by Lemma 2.6), then, by Proposition 1.1, one has that the determinant map d_E is injective and $\operatorname{Im}(d_E) = \operatorname{Im} H^0(\gamma)$. Hence, by Lemma 2.6, one gets $\operatorname{Im} H^0(\gamma) = W$. This concludes the proof. □

Remark 2.11. *The same conclusion holds when C is taken to be a smooth irreducible curve of genus $g \geq 2$ such that*

$$\deg(F|_C) = c_1(F) \cdot H^{n-1} \qquad \text{for any vector bundles } F \text{ on } X.$$

This is clearly true if C is a complete intersection of divisors in $|H|$.

Definition 2.12. *Assume that (X, L, H, r) is admissible. We set U to be the union of all the open dense subsets U_C defined in Lemma 2.9. We also set $\underline{c} = (c_1(L), c_1(L)^2, \cdots, c_1(L)^n)$.*

Consider now the moduli space parametrizing μ_H-semistable vector bundles E on X with rank r, $\det E \simeq L$ and Chern classes $c_i(E) = c_1(L)^i$, $i = 2, \ldots, n$.
Let $U \subset \mathrm{Gr}(r+1, H^0(L))$ the open subset defined in Definition 2.12. We have a map

$$U \to \mathcal{M}^s_H(r, L, \underline{c}) \qquad W \mapsto [E_W]. \tag{24}$$

Proposition 2.13. *The above map defines a rational map*

$$\Phi : \mathrm{Gr}(r+1, H^0(L)) \dashrightarrow \mathcal{M}^s_H(r, L, \underline{c}).$$

In particular, $\mathcal{M}^s_H(r, L, \underline{c})$ is not empty.

Proof. We prove that $\Phi|_U$ is a morphism. Let $\mathcal{U}$ and $\mathcal{Q}$ be the universal and quotient bundle on $G := \mathrm{Gr}(r+1, H^0(L))$. They fit into the exact sequence

$$0 \to \mathcal{U} \hookrightarrow H^0(L) \otimes \mathcal{O}_G \to \mathcal{Q} \to 0. \tag{25}$$

Consider the product $G \times X$ with its projection π_i on its factors. By pulling back along π_1 one has an inclusion $\pi_1^* \mathcal{U} \hookrightarrow H^0(L) \otimes \mathcal{O}_{G\times X}$. On the other hand, one can also pullback the evaluation map $ev_{H^0(L)} \otimes \mathcal{O}_X \to L$ along π_2. The composition θ of these maps gives a commutative diagram

$$\begin{array}{ccc} \pi_1^*(\mathcal{U}) & \hookrightarrow & H^0(L) \otimes \mathcal{O}_{G\times X} \\ & \searrow^{\theta} & \downarrow{\scriptstyle \pi_2^*(ev)} \\ & & \pi_2^*(L) \end{array}$$

In particular, $\theta|_{\{W\}\times X}$ is the evaluation map $ev_W : W \otimes \mathcal{O}_X \to L$. Hence, $\mathrm{Ker}(\theta)$ is locally free on the open set $U \times X$ by the above results. Then we can set

$$\mathcal{E} = \mathcal{H}om\left(\mathrm{Ker}(\theta)|_{U\times X}, \mathcal{O}_{U\times X}\right)$$

and observe that for all $W \in U$ one has

$$\mathcal{E}|_{\{W\}\times X} \simeq \mathrm{Ker}(W \otimes \mathcal{O}_X \to L)^* = E_W.$$

This implies that the map $U \to \mathcal{M}^s_H(r, L, \underline{c})$ such that $W \mapsto [E_W]$ is a morphism. □

We stress that, a priori, Φ could be defined on a bigger open subset containing U.

Theorem 2.14. *The restriction $\Phi \colon U \to \mathcal{M}^s_H(r, L, \underline{c})$ is an injective morphism. In particular, the moduli space $\mathcal{M}^s_H(r, L, \underline{c})$ contains a variety birational to $\mathrm{Gr}(r+1, H^0(L))$.*

Proof. By by Proposition 1.1 have a map

$$d \colon \Phi(U) \to \mathrm{Gr}(r+1, H^0(L))$$

sending $E \mapsto \mathrm{Im}(d_E)$, where d_E is the determinant map of E. By Proposition 2.10 we have that $d(\Phi(W)) = W$ for any $W \in U$. This implies that Φ is injective. In particular, by generic smoothness, the closure of $\Phi(U)$ is a variety birational to $\mathrm{Gr}(r+1, H^0(L))$). □

As a immediate consequence, $\mathcal{M}_H^s(r, L, \underline{c})$ has an irreducible component of dimension at least $(r+1)(d-g-r)$, by Remark 2.5.

3. Some examples for surfaces

In the previous section, we proved that, given an admissible collection (X, L, H, r), then the moduli space $\mathcal{M}_H^s(r, L, \underline{c})$ contains a subvariety birational to the Grassmanian $\mathrm{Gr}(r+1, H^0(L))$. In this section, we will present some examples of such collections when X is a smooth algebraic surface, denoted, from now on, by S. We will produce examples for every possible value of the Kodaira dimension.

The admissible data (S, L, H, r) will be presented with more details for the case of surfaces of general type and for surfaces of Kodaira dimension 0; for the other cases, we will not report most of the computations since they are similar to previous ones. Moreover, for the case of $K3$ surfaces, we will make a finer analysis and obtain Lagrangian subvarieties of the moduli space of sheaves (with suitable invariants).

3.1. **Surfaces of general type.** Let us assume that S is a minimal surface of general type with K_S ample. In particular, under these assumptions, S coincides with its canonical model. We also assume that S admits a smooth irreducible curve C in $|K_S|$. Notice that its genus is $g(C) = 1 + K_S^2 \geq 2$.

We observe that mK_S is ample for all $m \geq 1$. Nevertheless, it is not necessarily globally generated.

Remark 3.1. *As S is a canonical model, then, by results of Bombieri and Reider (see [Bom73] and [Rei88]), one has that mK_S is very ample (and so also globally generated) when*

$$m \geq 5 \;\; \text{if} \;\; K_S^2 \leq 2 \qquad \text{or} \qquad m \geq 3 \;\; \text{if} \;\; K_S^2 \geq 3. \tag{26}$$

Remark 3.2. *If we assume $m \geq 3$, one has that $(m-1)K_S$ and $(m-2)K_S$ are ample. Hence, for any $j \geq 1$, one has*

$$H^j(mK_S) = H^j(K_S + (m-1)K_S) = 0 \qquad H^j(mK_S - C)) = H^j(K_S + (m-2)K_S) = 0$$

by the Kodaira Vanishing Theorem. In particular, we have $H^1(mK_S - C) = 0$, so the restriction map $\rho : H^0(mK_S) \to H^0(mK_S|_C)$ is surjective. Moreover we have

$$h^0(mK_S) = \chi(mK_S) = \chi(\mathcal{O}_S) + \frac{m(m-1)}{2} K_S^2.$$

We set $H := K_S$ and $L := mK_S$, we want to find positive integers m and $r \geq 2$ such that (S, mK_S, K_S, r) is admissible and then Theorem 2.14 applies.

We observe that property A_1 is automatically satisfied by our assumptions on S. Before investigating property A_2, let us study the numerical properties A_3. Let us study separately what are the constraints on m and r for which either $A_3(1)$ or $A_3(2)$ holds, since our construction cannot be carried out for all pairs (r, m).

$A_3(1)$**:** The condition $\deg(L|_C) = rg(C) + 1$ holds if and only if

$$mK_S^2 = r(K_S^2 + 1) + 1. \tag{27}$$

Lemma 3.3. *Property $A_3(1)$ holds if and only if r and m satisfy one of the following necessary conditions:*

- $K_S^2 = 1$**:** *$r \geq 2$ and $m = 2r + 1$;*
- $K_S^2 = 2$**:** *$r = 2a - 1$ and $m = r + a = 3a - 1$ with $a \geq 2$.*

$K_S^2 \geq 3$: *$r = aK_S^2 - 1$ and $m = r + a = a(K_S^2 + 1) - 1$ with $a \geq 1$.*

Proof. If $K_S^2 = 1$ the condition $m = 2r + 1$ follows directly from Equation (27).
Assume now $K_S^2 \geq 2$. Reducing Equation (27) modulo K_S^2 one gets $r \equiv K_S^2 - 1 \mod K_S^2$, so that $r = aK_S^2 - 1$ for suitable $a \geq 1$. Then

$$(28) \qquad m = \frac{1}{K_S^2}\left[r(K_S^2+1)+1\right] = \frac{1}{K_S^2}\left[(aK_S^2-1)(K_S^2+1)+1\right] = aK_S^2 + (a-1) = r + a.$$

If $K_S^2 = 2$ one has $r = 2a - 1$ which satisfy the constrain $r \geq 2$ only if $a \geq 2$, so we have to exclude the case $a = 1$. □

$A_3(2)$: In this case, the condition is

$$(29) \quad r + K_S^2 + 2 \leq mK_S^2 \leq \min(2r, r + 2K_S^2 + 2), \quad \text{and if} \quad mK_S^2 = 2r,\ C \text{ is not hyperlliptic.}$$

For brevity, we consider the set

$$\mathcal{S}_{\bar r} = \bigcup_{r \geq \bar r} \{(r, m(r)), (r, m(r)+1) \,|\, r \equiv -2 \mod K_S^2\} \cup \{(r, \lceil m(r) \rceil) \,|\, r \not\equiv -2 \mod K_S^2\}$$

where we define $m(r) = 1 + \frac{r+2}{K_S^2}$.

Lemma 3.4. *Condition $A_3(2)$ holds if and only if the pair (r,m) falls in one of the following cases*

K_S^2	*sporadic pairs*	*standard pairs*
1	$(4,7)$	$\mathcal{S}_5$
2	$(4,4)^\dagger, (5,5)^\dagger, (6,6)^\dagger, (6,5)$	$\mathcal{S}_7$
3	$(6,4)^\dagger, (7,4)$	$\mathcal{S}_8$
4	$(6,3)^\dagger, (8,4)^\dagger, (10,5)^\dagger, (9,4), (10,4)$	$\mathcal{S}_{11}$
$K_S^2 \geq 5$ *odd*	$(4, 2K_S^2)^\dagger, \{(r,3) \mid \lceil 3K_S^2/2 \rceil < r \leq 2K_S^2 - 2\}$	$\mathcal{S}_{2K_S^2+1}$
$K_S^2 \geq 5$ *even*	$(3, 3K_S^2/2)^\dagger, (4, 2K_S^2)^\dagger, \{(r,3) \mid 3K_S^2/2 < r \leq 2K_S^2 - 2\}$	$\mathcal{S}_{2K_S^2+1}$

For those pairs marked with the symbol †, we also require that the general curve $C \in |H|$ is not hyperelliptic.

Proof. The pairs in the table are obtained by analysing the condition (29). When $K_S^2 = 1$, the pairs $(3,6)$ and $(4,8)$ were excluded since, the general element of $|K_S|$ is hyperelliptic. □

We can finally state and prove

Theorem 3.5. *Let S be a minimal surface of general type with ample canonical class. Assume that the pair (r,m) satisfies either the condition of Lemma 3.3 or Lemma 3.4 and that the canonical linear system $|K_S|$ contains a smooth irreducible curve. Then, with the possible exception of the case $K_S^2 = 2$ with $(r,m) = (4,4)$, (S, mK_S, K_S, r) is admissible and there exists a subvariety of $\mathcal{M}^s_{K_S}(r, mK_S, m^2K_S^2)$ birational to $\mathrm{Gr}(r+1, H^0(mK_S))$.*

Proof. As remarked above, property A_1 is automatically satisfied by assumptions on S. Instead, property A_3 holds by Lemmas 3.3 or 3.4. Finally, for all the above pairs with the exception, for $K_S^2 = 2$, of the pair $(4,4)$, the Inequalities (26) in Remark 3.1 hold so mK_S is very ample and $m \geq 3$. In particular, condition A_2 is automatically satisfied from Remark 3.2. Then the claim follows by applying Theorem 2.14. □

In order to apply Theorem 3.5 we need to check whether $|K_S|$ actually contains a smooth and irreducible element. Unfortunately, the existence of such a curve really depends on the family of surfaces we are considering and not only on numerical data on S.

If we know that $|K_S|$ is base-point free, then the problem is solved by Bertini's Theorem. However, the assumption we need, namely to pick up a smooth irreducible curve C in $|K_S|$, is weaker than to require $|K_S|$ base-point free.

Indeed, there are several examples of minimal surfaces of general type with ample canonical system with base points, but with a smooth irreducible canonical curve. For example, if $K_S^2 = 1$, one has Todorov's surfaces (see [Tod80]) for which $|K_S|$ has fixed part and some surfaces studied by Horikawa and Kodaira (see [Hor76]) for which $|K_S|$ has a single (simple) base point.

Remark 3.6. *By Reider's Theorem (see* [Rei88]*), the bicanonical map is always a morphism if $K_S^2 \geq 5$. This implies that the general bicanonical curve C is smooth and irreducible by Bertini. Thus, it is natural to construct other examples by setting $H := 2K_S$ and $L := mK_S$ as property A_1 always holds. Using similar computations to satisfy condition $A_3(1)$ as in the previous case, we obtain the following theorem.*

Theorem 3.7. *Let S be a minimal surface of general type with a ample canonical class and $K_S^2 \geq 6$, with K_S^2 even number. Given an even number $a \geq 2$, let us consider a pair (r, m) such that*

$$r = aK_S^2 - 1 \qquad \text{and} \qquad m = \frac{a(3K_S^2 + 1) - 3}{2}.$$

Then $(S, mK_S, 2K_S, r)$ is admissible and there exists a subvariety of $\mathcal{M}^s_{2K_S}(r, mK_S, m^2K_S^2)$ birational to $\mathrm{Gr}(r + 1, H^0(mK_S))$.

We point out that one can also try to get constrains on r and m in order to satisfy condition $A_3(2)$ instead of $A_3(1)$. In this case, we would get a similar theorem such as Theorem 3.7.

3.2. **Surfaces with Kodaira dimension** 0**.** Let us assume now that S is a smooth algebraic surface with $K_S \equiv_{num} 0$. Let us consider a very ample line bundle H on S. The assumption on the very ampleness of H puts a lower bound on the possible values of H^2, depending on the class of surfaces we are considering. For example, if S is a K3, one has $H^2 \geq 4$ with equality realised if and only if S is a smooth quartic in $\mathbb{P}^3$. If S is not a K3, one necessarily has $H^2 \geq 10$ (see [Mum08], [BPVdV84], [CDL25], for example). We recall, moreover, that H^2 is even since K_S is numerically trivial.

By assumption, a general curve $C \in |H|$ is smooth and irreducible of genus $g(C) = 1 + \frac{1}{2}H^2 \geq 2$, hence property A_1 is satisfied. We set $L := mH$, with $m \geq 2$. By Kodaira-vanishing, we also have that property A_2 holds.

Let's consider condition $A_3(1)$. As we set $L = mH$, the condition can be rewritten as

$$mH^2 = r\left(1 + \frac{1}{2}H^2\right) + 1. \tag{30}$$

It is easy to see that this condition implies that H^2 has to be a multiple of 4.

With similar computations as the ones done for surfaces of general type, we obtain the following result.

Lemma 3.8. *For brevity, we set $h := H^2/4$ with H very ample as above. The numerical condition $A_3(1)$ holds if and only if r and m can be written, for a given natural number $a \geq 1$*

as follows:

$$r = 4a + 1 \qquad \text{and} \qquad m = 3a + 1, \qquad \text{if } h = 1,$$

$$r = 4ha - 2h - 1 \qquad \text{and} \qquad m = (1 + 2h)a - h - 1, \qquad \text{if } h \geq 2.$$

Similarly, the condition $A_3(2)$ can be rewritten as

$$(31) \quad r + \frac{1}{2}H^2 + 2 \leq mH^2 \leq \min(2r, r + H^2 + 2), \quad \text{and if} \quad mH^2 = 2r, \ C \text{ is not hyperlliptic.}$$

We define an auxiliary set in order to describe in a more compact way the set of solutions. For a given $h \geq 2$ consider the following all $m, h \geq 2$ set

$$a_m = h(2m - 2) - 2 \qquad b_m = a_m + h = h(2m - 1) - 2$$

and observe that $a_{m+1} - b_m = h$ so that the intervals $[a_m, b_m]$ are all disjoint (and exactly $h+1$ integer can be found in any of these intervals). In analogy with what we have done for the case of surfaces of general type, we consider the set

$$\mathcal{T}_{\bar{m}} = \bigcup_{m \geq \bar{m}} \left([a_m, b_m] \cap \mathbb{Z}\right) \times \{m\}.$$

Lemma 3.9. *For brevity, we set $h := H^2/2$ where H is a very ample divisor. Then, the numerical condition $A_3(2)$ holds if and only if the pair (r, m) falls in one of the following cases:*

h	*sporadic pairs*	*standard pairs*
2	$(4,2), (6,3), (7,3), (8,3)$	$\mathcal{T}_4$
3	$(6,2), (7,2)$	$\mathcal{T}_3$
4	$(8,2), (9,2), (10,2)$	$\mathcal{T}_3$
≥ 5	$(2h,2)^\dagger, (2h+1,2), ([2h+2, b_2] \cap \mathbb{Z}) \times \{2\}$	$\mathcal{T}_3$

For those pairs marked with the symbol †, if K_S is not trivial, we also require that the general curve $C \in |H|$ is not hyperelliptic.

Proof. The pairs are obtained by analysing the condition (31). The first value of h to be considered is 2 since the minimum value of $H^2 = 2h$ for a very ample divisor on a surface with numerically trivial canonical bundle is 4. The condition $mH^2 = 2r$ (which is the case for which we need to check whether the general element of $|H|$ is not hyperelliptic) is satisfied only by the pairs

$$(6,3) \text{ for } h = 2 \qquad \text{and} \qquad (2h, 2) \text{ for } h \geq 2.$$

We claim that the only possible cases, among those, for which one could have that the general element of $|H|$ is hyperelliptic, appear at most for $h \geq 5$ and when K_S is not trivial. Indeed, if K_S is trivial and if C is a smooth element in $|H|$, the canonical divisor of C is given by $K_C = H|_C$. This implies that the restriction of the embedding $\varphi_{|H|}$ induces an embedding of C given by a subsystem of the canonical system. Then, C cannot be hyperelliptic when K_S is trivial. In order to conclude the proof, it is enough to remember, as recalled at the beginning of the subsection, that $h \geq 5$ if S is not a K3. □

Theorem 3.10. *Let us consider a surface S with $K_S \equiv_{num} 0$ and let H be a very ample line bundle. Assume that (r, m) satisfies either the conditions of Lemma 3.8 and 3.9. Then (S, mH, H, r) is admissible and there exists a subvariety of $\mathcal{M}^s_H(r, mH, (mH)^2)$ birational to $\mathrm{Gr}(r + 1, H^0(mH))$.*

It is actually useful to be more precise about the subvariety of $\mathcal{M}_H^s(r,L,L^2)$ which is birational to the Grassmannian $\mathrm{Gr}(r+1,H^0(L))$ in these cases. Indeed, if we can apply Theorem 2.14, then, we have an injective morphism

$$\Phi : U \to \mathcal{M}_H^s(r,L,L^2) \qquad W \mapsto E_W$$

where U is a dense open subset of $\mathrm{Gr}(r+1,H^0(L))$.
Note that, by Kodaira vanishing, we have $h^0(L)=\chi(L)=\chi(\mathcal{O}_S)+\frac{1}{2}L^2$, so we obtain

$$\dim(\mathrm{Gr}(r+1,H^0(L))) = (r+1)\left(\frac{1}{2}L^2+\chi(\mathcal{O}_S)-r-1\right). \tag{32}$$

We recall (see Section 1) that the moduli space $\mathcal{M}_H^s(r,L,L^2)$ is not empty and it is smooth of the expected dimension if $\mathrm{Ext}^2(E,E)_0=0$ for all $[E]\in\mathcal{M}_H^s(r,L,L^2)$, where

$$\begin{aligned}\mathrm{Ext}^2(E,E)_0 &= \ker\left(h^2(tr)\colon \mathrm{Ext}^2(E,E)\to H^2(\mathcal{O}_S)\right)\simeq\\ &\simeq \ker\left(h^2(tr)\colon \mathrm{Hom}(E,E\otimes K_S)^*\to H^0(K_S)^*\right),\end{aligned}$$

and the latter isomorphism follows from Serre duality. We end up in one of the two possible cases:

- If K_S is trivial, we have $\mathrm{Ext}^2(E,E)_0\simeq \mathrm{Hom}(E,E)_0^*=0$ as E is simple (see also [HL10, page 168]).
- If K_S is not trivial, then $H^2(\mathcal{O}_S)\simeq H^0(K_S)^*=0$ by Serre duality. Then, $\mathrm{Ext}^2(E,E)_0\simeq \mathrm{Ext}^2(E,E)\simeq \mathrm{Hom}(E,E\otimes K_S)^*$. Now, both E and $E\otimes K_S$ are H-stable vector bundles, with the same rank and the same H-slope: we have that $\mathrm{Hom}(E,E\otimes K_S)$ is trivial unless $E\simeq (E\otimes K_S)$. If this happens, then $c_1(E)=c_1(E)+rK_S$ so, K_S is a torsion line bundle whose order divides r. It is well known that the possible orders for K_S when $K_S\neq\mathcal{O}_S$ are $2,3,4$ and 6, with the last three cases occurring only when S is hyperelliptic (see [BPVdV84, page 188]).

Following the above reasoning, we can conclude that if S is a $K3$ or abelian surface then we have $\mathrm{Ext}^2(E,E)_0=0$, for any $[E]\in\mathcal{M}_H^s(r,L,L^2)$. So the moduli space is smooth and its dimension, given by Equation (2), is the following:

$$\dim(\mathcal{M}_H^s(r,L,L^2)) = (r+1)(L^2-(r-1)\chi(\mathcal{O}_S)). \tag{33}$$

Notice, in particular, that

$$\dim(\mathrm{Gr}(r+1,H^0(L)))\leq \frac{1}{2}\dim(\mathcal{M}_H^s(r,L,L^2))$$

with equality if and only if $\chi(\mathcal{O}_S)=2$, i.e. if and only if S is a $K3$ surface.

Theorem 3.11. *Let S be a $K3$ surface and let H be an ample primitive line bundle. If we choose (r,m) and L as in Theorem 3.10 then, whenever $\mathcal{M}_H(r,L,L^2)$ is a smooth irreducible symplectic variety, the closure of $\mathrm{Im}(\Phi)$ in $\mathcal{M}_H(r,L,L^2)$ is a (possibly singular) Lagrangian subvariety.*

Proof. Consider the Mukai vector $v=(r,L,r-L^2/2)=(r,mH,r-m^2\frac{H^2}{2})$ associated to our construction (see [HL10]). Then, with the notation in [PR23], we have $\mathcal{M}_H(r,L,L^2)=\mathcal{M}_v(S,H)$ and $\mathcal{M}_H^s(r,L,L^2)=\mathcal{M}_v^s(S,H)$. It is enough to recall that $Y=\overline{\mathrm{Im}(\Phi)}$ is birational to $\mathrm{Gr}(r+1,H^0(L))$ and thus it is rational. Hence, (non-zero) holomorphic 2-forms on $\mathcal{M}_v(S,H)$ restrict to a trivial 2-form on Y_{reg}. On the other hand, when S is a $K3$ one has $\chi(\mathcal{O}_S)=2$ and we have

$$\dim(\mathcal{M}_v(S,H))=\dim(\mathcal{M}_H^s(r,L,L^2))=2\dim(\mathrm{Gr}(r+1,H^0(L)))=2\dim(Y)$$

so Y is a Lagrangian subvariety of $\mathcal{M}_v(S,H)$. □

Remark 3.12. *If S has Picard number one, then $\mathcal{M}_H(r,L,L^2)$ is a smooth irreducible symplectic variety and Theorem 3.11 applies whenever (r,m) are coprime*[1]*. Indeed, since H is primitive, we have that the Mukai vector $v = \left(r, mH, r - m^2\frac{H^2}{2}\right)$ is primitive too. Since $\rho(S)=1$, then H is both v-generic and general with respect to v (see* [PR23, Lemma 2.9]*). Then, by* [PR23, Theorem 1.10] *(and see also* [KLS06, Theorem 4.4]*), $\mathcal{M}_v(S,H)$ is an irreducible simplectic variety. The smoothness follows from the fact that v is primitive and H is general with respect to v (see either the remark following Theorem 1.10 in* [PR23] *or* [Saw16, Lemma 2]*).*

A different explicit description of the Lagrangian subvariety of the moduli space of stable vector bundles on a smooth regular algebraic surface with $p_g > 0$ can be found in [YGY93].

3.3. **Del Pezzo surfaces.** Let us assume that S is a del Pezzo surface and let e be its degree (i.e. $e = K_S^2$). We recall (see [Dem76], for example) that, although $-K_S$ is ample, it is not very ample when $e \leq 2$. On the other hand, $-2K_S$ is always globally generated, and it is very ample unless $e=1$, whereas $-3K_S$ is always very ample. We set

$$L = -mK_S \quad H = -3K_S$$

with $m \geq 2$ so that L is nef, big and globally generated, and there exists a smooth irreducible curve C in the linear system $|H|$ (i.e. assumption A_1 holds). Moreover, as $L - H = (m-3)K_S$, it follows that $H^1(L-H) = 0$ (by Kodaira vanishing for $m \neq 3$ and since S is regular, for the case $m=3$), hence the restriction map of global section $\rho\colon H^0(L) \to H^0(L|_C)$ is surjective (so that assumption A_2 holds).

We would like to find pairs (r,m) such that assumption A_3 holds too for some integer $r \geq 2$. For this class of surfaces, for brevity, we focus only on assumption $A_3(1)$. Analogous results hold for the case $A_3(2)$ and can be easily obtained.

As $g(C) = 1 + 3K_S^2 = 1 + 3e \geq 4$, the condition $\deg(L|_C) = rg + 1$ is equivalent to $3me = r(1+3e)+1$. Then, one has necessarily

$$r = 3ae - 1 \quad \text{and} \quad m = a(3e+1) - 1, \qquad \text{with } a \geq 1. \tag{34}$$

Notice that, under our assumptions, we have no solution when $m = 2$. So we have the following result:

Theorem 3.13. *Let S be del Pezzo surface of degree e. For any natural number $a \geq 1$ we consider a pair (r,m) as in (34). Then, if $L = -mK_S$ and $H = -3K_S$, to the triple (S,L,H,r) is admissible and there exists a subvariety of $\mathcal{M}_H^s(r,L,L^2)$ birational to* $\mathrm{Gr}(r+1, H^0(L))$.

3.4. **Elliptic surfaces.** Here, we present two classes of examples of admissible data for elliptic surfaces, focusing specifically on assumption $A_3(1)$. These surfaces are of *product-quotient* type, a class that has been extensively investigated in the literature (see, e.g., the recent works [Fal24], [FGR26], and [AFG25]).

First of all, let us consider a surface $S = E \times F$ where E is an elliptic curve and F is a curve of genus $g \geq 2$ so that S is an elliptic surface. We write K_F to mean any canonical divisor on F. Hence, K_F is globally generated and ample and the same holds for the divisor $2p$ on E, where p is any point on E. Then, if we set $H = 2(p \times F) + E \times K_F$, we have that H is globally generated

[1] This actually happens for all the pairs satisfying condition $A_3(1)$, i.e. the ones given in Lemma 3.8. There are also pairs that satisfy condition $A_3(2)$ for which (r,m) are coprime.

and ample so that assumption A_1 holds by Bertini's Theorem. It is easy to see that any smooth curve C in $|H|$ has genus $g(C) = 6g - 5$.

If we set $L = mH$ with $m \geq 3$ we have $L - H \equiv K_S + D$ with D ample, so $H^1(L - H) = 0$ by Kodaira vanishing: assumption A_2 holds.

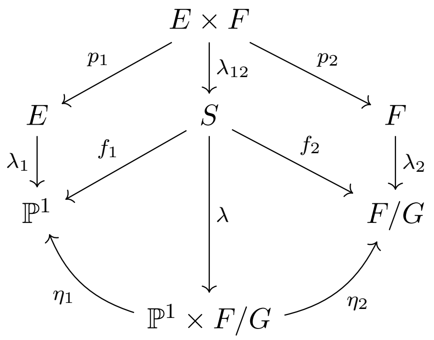
Reasoning as in the other cases, after some computation, one proves the following:

Theorem 3.14. *If $S = E \times F$ with E an elliptic curve and $g = g(F) \geq 2$, set $H = 2(p \times F) + E \times K_F$. For any integer $a \geq (7g - 3g^2)/(6g - 5)$ we consider a pair (r, m) such that*

$$r = 4g^2 - 10g + 5 + 8(g - 1)a \qquad \text{and} \qquad m = 3g^2 - 7g + 3 + a(6g - 5).$$

Then, if we set $L = mH$, then (S, L, H, r) is admissible and there exists a subvariety of $\mathcal{M}^s_H(r, L, L^2)$ birational to $\mathrm{Gr}(r + 1, H^0(L))$.

For the second class of examples, we slightly modify the previous one. Let us consider a finite group G acting faithfully both on an elliptic curve E and a smooth curve F of genus $g \geq 2$, such that $E/G \cong \mathbb{P}^1$ and F/G is an elliptic curve. Assume that the action of the diagonal subgroup $\Delta \leq G \times G$ on the product $E \times F$ is free, so that the quotient $S := (E \times F)/\Delta$ is smooth. Surfaces of this type are said to be *isogenous to a product of curves* (see [Cat00, Def. 3.1]). We have the following hexagonal commutative diagram



involving projections from products (namely p_1, p_2, η_1 and η_2), quotients by the various actions of G (namely λ_1, λ_2 and λ_{12}) and the natural fibrations f_1 and f_2 induced on S.

Consider $p \in \mathbb{P}^1$ and $q \in F/G$ and the fibers $F_1 := f_1^*(p)$ and $F_2 := f_2^*(q)$. Clearly, the choice of the two points doesn't matter if we are only interested in the numerical class of F_1 and F_2. We observe that S has Kodaira dimension one as $\lambda_{12}\colon E \times F \to S$ is a finite étale morphism of smooth surfaces. The numerical class of a canonical divisor of S is

$$K_S \equiv_{num} \frac{2g - 2}{|G|} F_2.$$

We observe that any irreducible curve C of S such that $C \cdot F_1 = 0$ is contained in a fiber of f_1; otherwise, we could always pick up a point of C such that the fiber of that point and C intersect positively. A similar argument holds when $C \cdot F_2 = 0$.

Let us consider a divisor $H := F_1 + 2F_2$. Since F_1 and F_2 are nef divisors (as f_1 and f_2 are fibrations), then H is ample by the previous argument. Indeed, $C \cdot H \geq 0$ with equality if and only if both $C \cdot F_1$ and $C \cdot F_2$ are zero, a contradiction. The divisor H is also globally generated as F_1 and $2F_2$ are globally generated as pullback, via a dominant morphism λ, of the globally generated divisor $\{p\} \times (F/G) + 2\mathbb{P}^1 \times \{q\}$. In particular, there exists a smooth curve $C \in |H|$ and A_1 holds. It is easy to see that the genus of any smooth curve $C \in |H|$ is $g(C) = 2|G| + g$.

If we set $L := mH$ with $m \geq 1 + (g - 1)/|G|$, then $L - H - K_S$ is ample, so that $H^1(L - H)$ is zero by Kodaira vanishing and A_2 hold.

Theorem 3.15. *Let $S = (E \times F)/G$ isogenous to a product of a an elliptic curve E and a curve F with $g = g(F) \geq 2$. Set $H = F_1 + 2F_2$ as above. For any integer a, consider the pair (r, m) with*

$$r = \frac{2(2a+1)|G| - 1}{g} \qquad \text{and} \qquad m = \frac{(2a+1)(2|G| + g) - 1}{2g}.$$

Then, if r and m are integers with $r \geq 2$ and $m \geq 2$, and if we set $L = mH$, then (S, L, H, r) is admissible so there exists a subvariety of $\mathcal{M}^s_H(r, L, L^2)$ birational to $\mathrm{Gr}(r+1, H^0(L))$.

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